Renormalised perturbation theory of ordered systems



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# Renormalised perturbation theory of ordered systems

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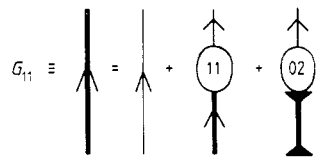
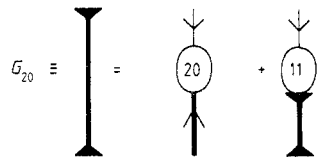
**Abstract.** We introduce a new self-consistent summation procedure for the analysis of Feynman–Dyson perturbation series and demonstrate how a self-consistent expression for the self-energies of an interacting system may be obtained. We obtain the simplest self-consistent solution to the problem of an interacting many fermion system exhibiting long-range order, characterised by the existence of anomalous propagators, in which correlation effects not described by the Hartree–Fock–Gorkov approximation are important.

## 1. Introduction

Conventional perturbation theory expresses the single particle Green function $G_{11}$ of a system of fermions interacting via an instantaneous, two-body potential $U$, in terms of an infinite series of connected diagrams. In standard notation (Fetter and Walecka 1971, Mattuck and Johansson 1968) the description of a system, which exhibits long-range order characterised by the appearance of anomalous propagators ($G_{02}$ and $G_{20}$) below a transition temperature, is given by equations (1.1)–(1.3)




(1.1)

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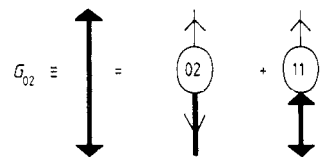


(1.3)

where a thin solid line running from $y$ to $x$ represents a single particle Green function $g(x, y)$ of the non-interacting system.

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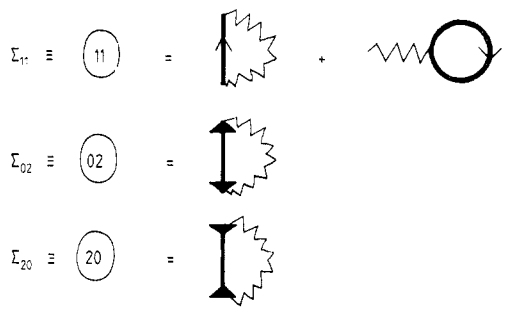
Denoting the potential $(i/\hbar)U(x, y)$ by a wavy line joining the points $x$ and $y$, the first-order contributions to the self-energies are shown explicitly in equations (1.4)




(1.4)

These diagrams, when coupled with equations (1.1)–(1.3), represent an extension of the Hartree–Fock approximation to the description of systems exhibiting long-range order and form the basis of the Gorkov (1958) formulation of the theory of superconductivity.

The success of this first-order approximation is arguably a consequence of the self-consistency imposed by the presence of the exact propagators of the interacting system, on the right-hand side (RHS) of equations (1.4). However, when the interaction is singular, the above approximation will fail and in order to replace $U$ by a less singular, renormalised interaction, an infinite subset of the diagrams appearing on the RHS of equations (1.4) must be summed. Such a renormalisation procedure may be viewed as introducing a new degree of self-consistency into the solution of equations (1.1)–(1.3), which is not present in the first-order diagrams shown explicitly in equations (1.4).

The particular choice of which subset of higher-order diagrams to sum in a given situation is governed by the physics of the problem and such calculations invariably proceed in a rather *ad hoc* manner. Recently however, a self-consistent summation technique for the analysis of Feynman–Dyson perturbation series was employed to renormalise the interaction lines appearing in the perturbation theory of normal many fermion systems systematically (where $G_{02} = G_{20} = 0$) and it was demonstrated that the first-order terms of the resulting perturbation series were sufficient to describe most normal systems of physical interest (Lambert and Hagston 1981).

The success of this renormalisation technique has led us to consider, in the present paper, its application to the perturbation theory of systems exhibiting long-range order characterised by the existence of anomalous propagators. In particular, we provide a systematic evaluation of the 'first-order' approximation to equations (1.1)–(1.3) which incorporates the new degree of self-consistency, not present in the diagrams shown explicitly in equations (1.4).

## 2. Formulation of the problem

We begin our analysis by formulating the problem from the viewpoint of an infinite hierarchy of equations for the Green functions of the interacting system. Depending on the definitions of the various propagators (Mattuck and Johansson 1968), equations (1.1)–(1.4) may describe, for example, a superconducting electron gas or a ferromagnetic spin system and since much of the analysis described below follows straightforwardly from the topology of the perturbation series, the same will be true of our final result. However, for convenience we shall employ the following definitions for the

propagators of the system

$$(\mathrm{i})^{(m+n)/2}G_{m,n}(1,2,\ldots,m;1',2',\ldots,n')=\langle T[\psi_{(1)}\psi_{(2)}\ldots\psi_{(m)}\psi^{+}_{(1')}\psi^{+}_{(2')}\ldots\psi^{+}_{(n')}]\rangle \tag{2.1}$$

where $\frac{1}{2}(m+n)$ will always be an integer in what follows. The average $\langle\ldots\rangle$ in equation (2.1) could also be left unspecified in the following, because the analysis may be carried out at finite temperatures as well as $T=0$ K. However, again for convenience, we shall define our average with respect to the ground state $|\Psi\rangle$ of the system, which satisfies

$$H|\Psi\rangle=E|\Psi\rangle.$$

In this case, the Heisenberg picture field operators in equation (2.1) have the form

$$\psi_{(1)}\equiv\psi_{(\boldsymbol{x}_1,\,t_1)}=\exp(\mathrm{i}Ht_1/\hbar)\psi_{(\boldsymbol{x}_1)}\exp(-\mathrm{i}Ht_1/\hbar) \tag{2.2}$$

where, for a uniform, translationally invariant system,

$$\psi_{(\boldsymbol{x}_1)}=\sum_{k=0}^{\infty}V^{-1/2}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_1)a_k. \tag{2.3}$$

Writing

$$H=H_0+H_1 \tag{2.4}$$

where

$$H_0=\sum_k\hbar\omega_k a_k^{+}a_k \qquad \text{with} \qquad \omega_k=\frac{\hbar k^2}{2m}-\frac{\mu}{\hbar} \tag{2.5}$$

and

$$H_1=\tfrac{1}{2}\int \mathrm{d}^3x_1\,\mathrm{d}^3x_1'\,U(\boldsymbol{x}_1-\boldsymbol{x}_1')\psi^{+}_{(\boldsymbol{x}_1)}\psi^{+}_{(\boldsymbol{x}_1')}\psi_{(\boldsymbol{x}_1')}\psi_{(\boldsymbol{x}_1)} \tag{2.6}$$

allows us to write the following expression for the propagators defined by equation (2.1),

$$\begin{aligned}G_{m,n}(1,\ldots,m;1',\ldots,n')&=\sum_{j=1}^{n}(\pm1)^{j+1}g_{(m,j')}G_{m-1,n-1}(1,\ldots,m-1;1',\ldots,[j-1]',[j+1]',\ldots,n')\\&+\lim_{\varepsilon\to0^{+}}\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t_z\exp(-\varepsilon|t_z|)\int\mathrm{d}^3x_z\,\mathrm{d}^3x_z'\,g_{(m,z)}U(\boldsymbol{x}_z-\boldsymbol{x}_z')\\&\times G_{m+1,n+1}(1,\ldots,m-1,z',z;z',1',\ldots,n').\end{aligned} \tag{2.7}$$

The first term on the RHS of this equation vanishes for $n=0$ and equation (2.7) is valid for $m\neq0$. (An equation for the case where $m=0$ is obtained by taking the Hermitian conjugate of equation (2.7) with $n=0$.)

The factor $(-1)^{j+1}((+1)^{j+1})$ which applies for fermions (bosons) is readily accounted for by the Feynman rules which apply to the perturbation series we are about to obtain (Mattuck and Johansson 1968) and will therefore be ignored in what follows.

The hierarchy of equations defined by equation (2.7) forms a convenient starting point for the formulation of the many-body problem, not least because it lends itself readily to diagrammatic analysis. The diagram rules which are applicable to equations of this type have been discussed in detail elsewhere (Lambert and Hagston 1981), and

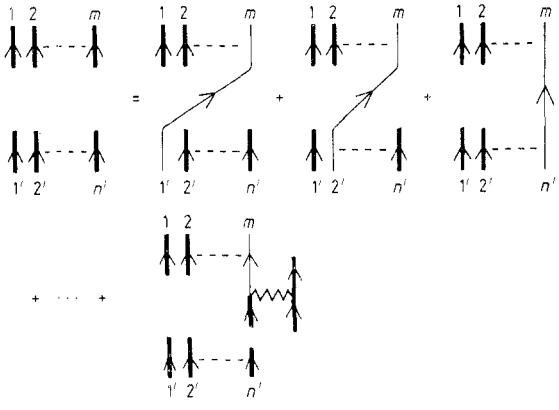
are essentially of the following form:

(i) Represent the propagator $G_{m,n}(1, 2, \ldots, m; 1', 2', \ldots, n')$ by $m(n)$ thick full lines with free ends entering (leaving) the points $1, 2, \ldots, m$ $(1', 2', \ldots, n')$.

(ii) Denote $g(j, j')$ by a thin full line running from $j'$ to $j$.

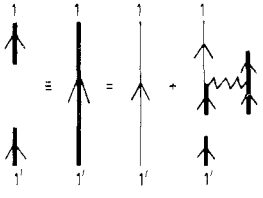
(iii) Denote $(\mathrm{i}/\hbar)U(\boldsymbol{x}_z - \boldsymbol{x}'_z)$ by a wavy line joining $\boldsymbol{x}_z$ and $\boldsymbol{x}'_z$.

These allow equation (2.7) to be written in the form of equation (2.8)

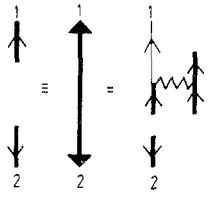


(2.8)

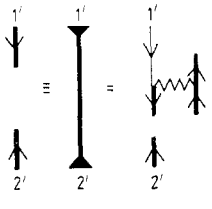
As a special case of the above rule, we shall employ the notation of equations (1.1)–(1.3) for the propagators $G_{11}$, $G_{02}$, $G_{20}$. The simplest of the above hierarchy of equations then take the form of equations (2.9)–(2.11)



(2.9)



(2.10)



(2.11)

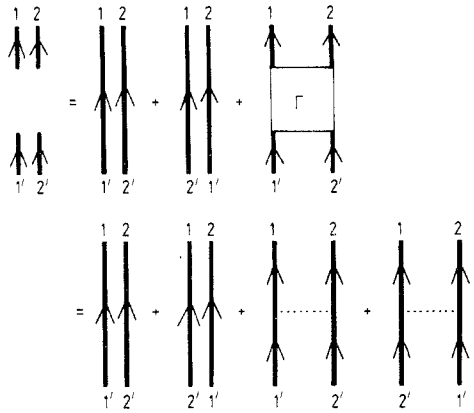
where we note that the latter is obtained from the Hermitian conjugate of equation (2.10).

The higher-order propagators appearing on the RHS of equations (2.9)–(2.11) may be evaluated using the techniques of conventional perturbation theory (Fetter and Walecka 1971, Mattuck and Johansson 1968) and the result is equations (1.1)–(1.4).

Thus in isolation, equations (2.9)–(2.11) represent no advantage over the conventional formulation of perturbation theory. However, when coupled with the higher-order equations defined by equation (2.7), the present formulation of the problem becomes extremely advantageous, because it allows us to obtain a *self-consistent* expression for the self-energies without explicitly summing terms in the original perturbation series. In order to illustrate this feature, we consider first how the higher-order equations may be employed to yield an exact expression for the self-energy $\Sigma_{11}$ in the absence of long-range order and then generalise the method to account for the presence of anomalous propagators. Our aim then, in the following section, is to obtain an exact expression for the various self-energies which occur when we deal with systems exhibiting long-range order.

## 3. The two particle propagator

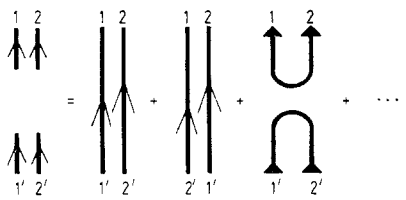
Above the transition temperature (where $G_{m,n} = 0$ unless $m = n$), conventional perturbation theory (Fetter and Walecka 1971, Nozieres 1964) shows that the two particle propagator $G_{2,2}(1, 2; 1', 2')$ may be written in the form of equation (3.1).




(3.1)

One half of the contribution to the term involving the vertex function $\Gamma$ may be obtained from the other half by simply permuting the labels 1′ and 2′ and, in equation (3.1), we have taken advantage of this feature to define a new vertex function $\Gamma'$, denoted by a broken line. This expression may be substituted into equation (2.9) to yield an exact expression for the self-energy $\Sigma_{11}$ (in the absence of long-range order) and in an earlier paper (Lambert and Hagston 1981) we demonstrated how this expression might be evaluated self-consistently.

Below the transition temperature, an inspection of the perturbation series (Mattuck and Johansson 1968) reveals that equation (3.1) must be generalised to equation (3.2) where the broken lines represent vertex function corrections.




(3.2)

The analysis is complicated by the fact that instead of just the single vertex function $\Gamma'$ (where one full line enters and one full line leaves each end of the corresponding broken line) there now appear six vertex functions. We shall denote these functions by $\alpha_1, \ldots, \alpha_6$ and employ the diagrammatic representation of figure 1.

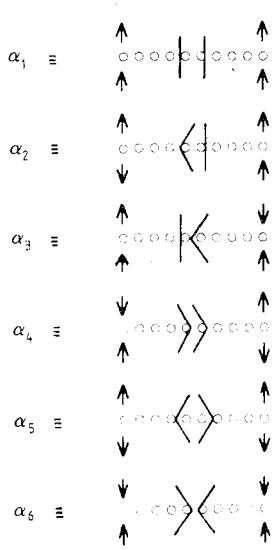
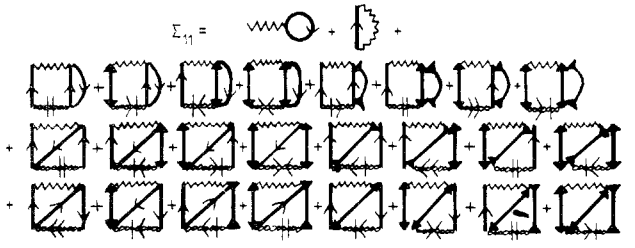
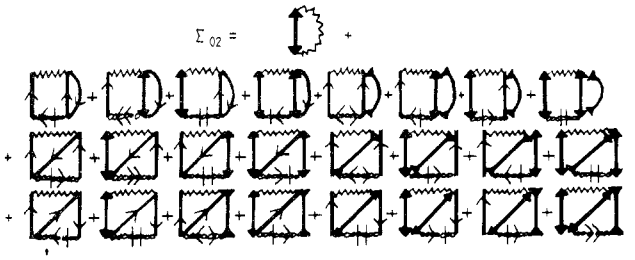


**Figure 1.**

With these definitions the RHS of equation (3.2) may be written explicitly in terms of the $\{\alpha_i\}$. Substituting the result into the RHS of equation (2.9) and comparing with equation (1.1) yields equations (3.3) and (3.4) for the self-energies $\Sigma_{11}$ and $\Sigma_{02}$ respectively. An expression for $\Sigma_{20}$ may be obtained by reversing the arrows in equation (3.4) and thus we obtain exact expressions



(3.3)



(3.4)

for the self-energies in the presence of long-range order. These equations form the

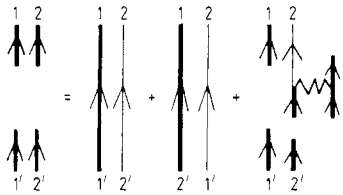
result which we have been seeking and express the self-energies in terms of the vertex functions $\{\alpha_i\}$. All that remains is for us to obtain an expression for the vertex functions and we carry out this task in the following section.

## 4. A self-consistent evaluation of the vertex functions

The method which we now introduce expresses the function $\{\alpha_i\}$ in terms of a highly summed perturbation series. However, previous calculations (Lambert and Hagston 1981) on systems where $G_{02} = G_{20} = 0$ have shown that terms beyond first order in the vertex function(s) are unimportant in many physical situations, and hence in the present analysis we shall restrict ourselves to the approximation of retaining terms which are at most linear in the $\{\alpha_i\}$.

The essential steps are as follows.

(i) In the previous section, we obtained exact expressions for the self-energies after evaluating the RHS of equation (2.9). To this end, we examined the perturbation expansion for $G_{2,2}$ (equation (3.2)) and obtained a closed form for $G_{2,2}$ in terms of the vertex functions $\{\alpha_i\}$. This result was then substituted into the RHS of equation (2.9) to yield, after comparison with equations (1.1)–(1.3), the desired expressions for the self-energies. The closed form for $G_{2,2}$ obtained in this way is exact and may be compared with an alternative expression, which we obtain as follows.




(4.1)

(ii) The expression for $G_{2,2}$ given by equation (2.8) is shown in equation (4.1) and involves a three particle Green function $G_{3,3}$. Following the method of the preceding section, we may express $G_{3,3}$ in terms of the vertex functions $\{\alpha_i\}$. However, unlike the expression for $G_{2,2}$ of step (i), the resulting equation for $G_{3,3}$ has the form of a series expansion in powers of the functions $\{\alpha_i\}$. Retaining terms up to first order in the vertex functions, we may substitute the result into equation (4.1) to obtain an alternative expression for $G_{2,2}$.

(iii) The equations for $G_{2,2}$ obtained in steps (i) and (ii) may now be compared to yield a self-consistent set of equations for the vertex function $\{\alpha_i\}$. The terms in the ensuing analysis have been carefully enumerated (Lambert 1979) and the result is given in figure 2, where $\alpha_6$ may be obtained from $\alpha_5$ by reversing the arrows.

These equations when coupled with equations (1.1)–(1.3) and (3.3)–(3.4) form a well defined theory from which many-body calculations on systems exhibiting long-range order and interacting via a singular potential might proceed.

## 5. Conclusion

The present analysis forms a systematic renormalisation of the time-dependent perturbation theory of systems exhibiting long-range order characterised by the

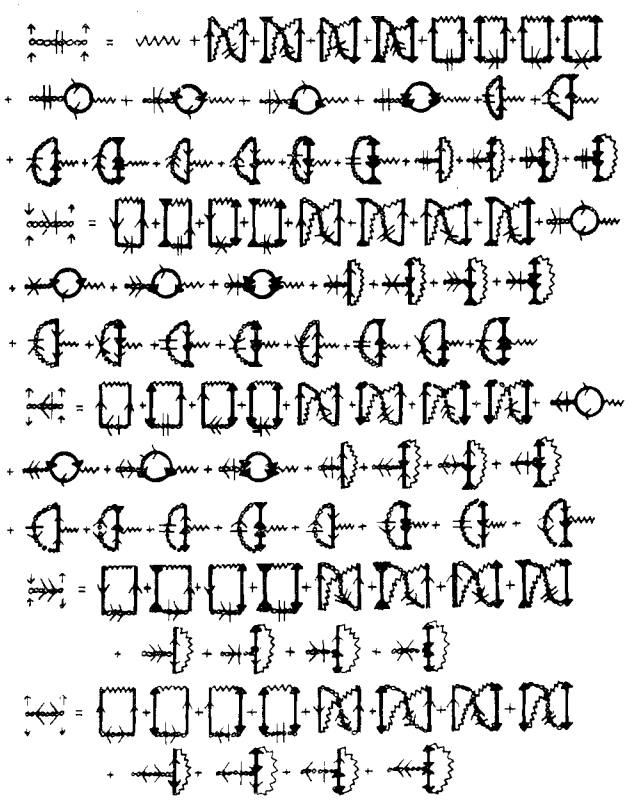


**Figure 2.**

existence of anomalous propagators and represents the only theory to date which can account for vertex function corrections in a thoroughly self-consistent manner. There are several new and important features present in the above formalism, not the least of which is the provision of a link between the well known equation of motion method (which deals with the differential equation equivalent of our hierarchy of equations) and infinite-order perturbation theory.

In a previous paper (Lambert and Hagston 1981), we utilised this connection to obtain a self-consistent perturbation theory for the description of correlation effects not accounted for by the Hartree–Fock approximation, and demonstrated that first-order terms in the resulting self-consistent expansion for the vertex function were sufficient to describe most normal systems of physical interest. In the present paper, we have extended this analysis to encompass systems which exhibit long-range order and derived a closed form for the self-energies in terms of the various vertex functions of the problem. This expression (shown in equations (3.3) and (3.4)) is exact and thus forms a useful starting point from which practical many body calculations may proceed.

In conventional perturbation theory, such a calculation would invariably proceed by performing the summation of an infinite subset of contributions to the various vertex functions, selected on the basis of physical arguments. However, in the present paper we have shown how this task may actually be avoided. By utilising higher-order equations in our hierarchy, we have demonstrated how a highly summed and manifestly self-consistent expression for the vertex functions may be obtained, without explicitly

summing diagrams in the original perturbation series, and in view of the fact that the linear terms in the resulting perturbation series are expected to be sufficient to describe most systems of physical interest, we have calculated these terms explicitly.

The result is summarised in figure 2, which combines with equations (3.3), (3.4) and (1.1)–(1.3) to yield a set of coupled integral equations. This set of self-consistent simultaneous equations represents the culmination of all the essential results of diagrammatic perturbation theory and any practical calculation is now reduced to the task of obtaining their solution. As an example of such a solution, we note that on ignoring vertex function corrections to the self-energies in equations (3.3) and (3.4), we obtain the celebrated Hartree–Fock–Gorkov approximation. As a less trivial example, we note that in the case of a dilute condensed Bose gas, interacting via a singular hard sphere potential, it has been shown that (Beliaev 1958) the dominant contributions to figure 2 arise from the ladder diagrams and a solution to the set of simultaneous equations is readily obtained. However, although this approximation has been successful in accounting for the long-wavelength (i.e. linear) region of the excitation spectrum of a superfluid Bose system, there has been no such success in accounting for the more complicated roton spectrum. We suggest that one reason for this is the large number of diagrams which must be accounted for in the conventional form of perturbation theory, as summarised in equations (1.1)–(1.3), when vertex function corrections are important. Since our final set of equations summarises these infinite sums of diagrams in a finite number of terms, we expect the present formalism to have important applications in this area of physics.

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